The Determination of the Cumulative Distribution Function and the Probability Density Function of Building Entry Loss based on ITU-R P.2109

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TERMS, CONDITIONS & NOTICES

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Preface

This document presents Probability Density Function (PDF) and Cumulative Distribution Function (CDF) curves based on the ITU-R P.2109 BEL model that can be considered in further discussions on AFC development. This material is meant to serve as technical background for those discussions. The document does not make any recommendations as to how an AFC system should operate or what specific methodologies or parameters should be used in its calculations.

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1 Introduction

Different methods can be used to predict Building Entry Loss (BEL), including in-situ measurement based techniques. The ITU-R P.2109 method is mentioned in the Report and Order FCC 20-51A1, April 2020 – see, for example, paragraphs 117-118, 122 and footnotes 297, 301, 465 in that document. The R&O also suggests that a mix of 70% traditional and 30% thermally efficient building types be used when determining BEL statistics. The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) for such a mixture can be constructed from the BEL PDF and CDF for each building type. However, the ITU-R P.2109 recommendation only provides an equation for the BEL inverse CDF, it does not provide a BEL CDF or PDF. The scope of this TR is to construct a composite BEL CDF/PDF for a mix of two different building types. This document describes a semi-analytic method that can be used to derive BEL CDF/PDF functions for a single building type based on the ITU-R P.2109 model. These can then be combined to create the composite CDF/PDF. The document also analyzes the dependence of the derived BEL CDF/PDF on frequency and elevation angle.

2 The ITU-R P.2109 Recommendation

The ITU-R P.2109 Recommendation provides a method to compute the Building Entry Loss (BEL) not exceeded for a specified probability. This computation requires the operating frequency, outdoor radiation elevation angle and building type to be specified along with the desired probability. Valid operating frequencies range from about 80 MHz to 100 GHz. No guidance is provided on the valid range of elevation angles. The building type can be either “traditional” or “thermally-efficient” construction. The recommendation does make some comments on the nature of each type of construction. Although the model supports probabilities from 0% to 100%, it should be noted that the model has only been validated against empirical data for probabilities from 1% to 99%.

3 Cumulative Distribution Function

This section presents equations for the BEL CDF in terms of parameters defined in P.2109.

3.1 Analytic Problem Definition

The BEL Cumulative Distribution Function (CDF) provides a probability P for a given loss value L not exceeded, or $P(L)$. This is the inverse of the function specified in P.2109 equation (1). The remainder of this section discusses how to compute the BEL CDF based on the relations in P.2109. The function $F(z)$ referred to in ITU-R P.2109 is the cumulative normal distribution function defined by the following integral. Using the substitution $x = t\sqrt{2}$ in the equation below, it can be related to the complementary error function as shown.

$$F(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx = 1 - \frac{1}{\sqrt{2\pi}} \int_{z}^{+\infty} e^{-x^2/2}\, dx = 1 - \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{2}}^{+\infty} e^{-t^2}\, dt = 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)$$

Where (see Abramowitz and Stegun eq. 7.1.2)

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2}\, dt$$

In P.2109 equations (2) and (3), values for $A(P)$ and $B(P)$ are computed using the inverse function $F^{-1}(P)$ such that:

$$A(P) = \sigma_1 F^{-1}(P) + \mu_1 \qquad B(P) = \sigma_2 F^{-1}(P) + \mu_2$$

And consequently

$$F\left(\frac{A - \mu_1}{\sigma_1}\right) = P \qquad F\left(\frac{B - \mu_2}{\sigma_2}\right) = P$$

Since $F(z)$ is a monotonic function it follows that for a given probability:

$$\frac{A(P) - \mu_1}{\sigma_1} = \frac{B(P) - \mu_2}{\sigma_2}$$

If one defines

$$Q(P) = \sigma_2[A(P) - \mu_1] = \sigma_1[B(P) - \mu_2]$$

Then

$$P = F\left(\frac{Q}{\sigma_1\sigma_2}\right) \qquad A(P) = \frac{Q}{\sigma_2} + \mu_1 \qquad B(P) = \frac{Q}{\sigma_1} + \mu_2$$

Furthermore, using the relation $10^{0.1x} = e^{K_0 x}$, equation (1) in P.2109 can be expressed as:

$$K_0 L(P) = \ln\left[e^{K_0 A(P)} + e^{K_0 B(P)} + K_1\right]$$

Where

$$K_0 \equiv 0.1 \ln(10) \approx 0.2302585093 \qquad K_1 \equiv e^{-3K_0} \approx 0.5$$

Hence:

$$e^{K_0 L} - K_1 = e^{K_0 A(P)} + e^{K_0 B(P)} = e^{K_0(Q/\sigma_2 + \mu_1)} + e^{K_0(Q/\sigma_1 + \mu_2)} > 0$$

Note that as the left hand side of the above equation must be positive, only values of $L > -3$ are valid. Specifying a loss value L determines a value for Q as all other parameters are known for a given frequency, elevation angle and building type. However, no simple closed form expression for Q exists, so some numerical method of solution or approximation is necessary. Once Q is found, the probability P can then be computed using the previous relation $P = F(Q/\sigma_1\sigma_2)$.

3.2 Numerical Solution

The relation

$$e^{K_0 L} - K_1 = e^{K_0(Q/\sigma_2 + \mu_1)} + e^{K_0(Q/\sigma_1 + \mu_2)} > 0$$

Can be recast as:

$$1 = M_1 e^{\alpha_2 Q} + M_2 e^{\alpha_1 Q}$$

Where

$$M_1 \equiv \frac{e^{K_0 \mu_1}}{e^{K_0 L} - K_1} \qquad \alpha_2 \equiv \frac{K_0}{\sigma_2} \qquad M_2 \equiv \frac{e^{K_0 \mu_2}}{e^{K_0 L} - K_1} \qquad \alpha_1 \equiv \frac{K_0}{\sigma_1}$$

and both M_1 and M_2 are positive. If one defines an objective function as:

$$f(x) \equiv \ln(M_1 e^{\alpha_2 x} + M_2 e^{\alpha_1 x}) \qquad f(Q) = 0$$

Then Newton-Raphson iteration can be used to find the value of x that drives the objective function to zero, which is the solution for Q. However, this technique does require an initial value close enough to the solution to converge. Two candidate initial values will be considered. If one or the other term in the objective function log argument sum dominates, then Q is approximately:

$$Q \approx x_A \equiv -\ln(M_1)/\alpha_2 \quad \text{or} \quad Q \approx x_B \equiv -\ln(M_2)/\alpha_1$$

Note that

$$f(x_A) = \ln(1 + M_2 e^{\alpha_1 x_A}) \qquad f(x_B) = \ln(M_1 e^{\alpha_2 x_B} + 1)$$

Consequently, the better initial value will correspond to the smaller of $M_2 e^{\alpha_1 x_A}$ or $M_1 e^{\alpha_2 x_B}$, since both terms are positive.

Furthermore, since α_1 and α_2 are also positive, it follows that if:

$$\begin{gathered} M_2 e^{\alpha_1 x_A} < M_1 e^{\alpha_2 x_B} \\ \ln(M_2) + \alpha_1 x_A < \ln(M_1) + \alpha_2 x_B \\ -\alpha_1 x_B + \alpha_1 x_A < -\alpha_2 x_A + \alpha_2 x_B \\ (\alpha_1 + \alpha_2) x_A < (\alpha_1 + \alpha_2) x_B \\ x_A < x_B \end{gathered}$$

Hence the initial value x_0 can be chosen as:

$$x_0 = \begin{cases} x_A; & x_A < x_B \\ x_B; & x_B \le x_A \end{cases}$$

Once an initial value has been determined, the iterative Newton-Raphson procedure is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{(M_1 e^{\alpha_2 x_k} + M_2 e^{\alpha_1 x_k}) \ln(M_1 e^{\alpha_2 x_k} + M_2 e^{\alpha_1 x_k})}{\alpha_2 M_1 e^{\alpha_2 x_k} + \alpha_1 M_2 e^{\alpha_1 x_k}}$$

Since

$$f(x) \equiv \ln(M_1 e^{\alpha_2 x} + M_2 e^{\alpha_1 x}) \qquad f'(x) \equiv \frac{df}{dx} = \frac{\alpha_2 M_1 e^{\alpha_2 x} + \alpha_1 M_2 e^{\alpha_1 x}}{M_1 e^{\alpha_2 x} + M_2 e^{\alpha_1 x}}$$

Although this numerical technique is certainly not the only method that can be used to find Q, it does seem to be stable and rapidly convergent for valid loss values. Typically, the iterative process will converge to double precision accuracy in three or four iterations. Once enough iterations have been computed so that x_k has converged to Q, the probability can be calculated from:

$$P = F\left(\frac{Q}{\sigma_1 \sigma_2}\right) = 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{Q}{\sigma_1 \sigma_2 \sqrt{2}}\right) = \frac{1}{2}\operatorname{erfc}\left(-\frac{Q}{\sigma_1 \sigma_2 \sqrt{2}}\right)$$

So, for any loss value $L > -3$ a corresponding probability $P(L)$ can be found. This $P(L)$ function is the Cumulative Distribution Function (CDF) for the Building Entry Loss (BEL).

4 Probability Density Function

Another function of interest is the BEL Probability Density Function (PDF), which is the derivative of $P(L)$ with respect to L. Since $P(L) = F(Q/\sigma_1\sigma_2)$, this function is:

$$p(L) \equiv \frac{dP}{dL} = \frac{1}{\sigma_1 \sigma_2} F'\left(\frac{Q}{\sigma_1 \sigma_2}\right) \frac{dQ}{dL}$$

Where by the fundamental theorem of calculus

$$F'(z) \equiv \frac{dF}{dz} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

And from implicit differentiation of $e^{K_0 L} - K_1 = e^{K_0 \mu_1} e^{\alpha_2 Q} + e^{K_0 \mu_2} e^{\alpha_1 Q}$ it follows that:

$$K_0 e^{K_0 L} = [\alpha_2 e^{K_0 \mu_1} e^{\alpha_2 Q} + \alpha_1 e^{K_0 \mu_2} e^{\alpha_1 Q}] \frac{dQ}{dL}$$

$$\sigma_1 \sigma_2 e^{K_0 L} = [\sigma_1 e^{K_0 \mu_1} e^{\alpha_2 Q} + \sigma_2 e^{K_0 \mu_2} e^{\alpha_1 Q}] \frac{dQ}{dL}$$

$$\sigma_1 \sigma_2 e^{K_0 L} = [\beta_1 e^{\alpha_2 Q} + \beta_2 e^{\alpha_1 Q}] \frac{dQ}{dL}$$

Where

$$\beta_1 \equiv \sigma_1 e^{K_0 \mu_1} \qquad \beta_2 \equiv \sigma_2 e^{K_0 \mu_2}$$

So that

$$\frac{dQ}{dL} = \frac{\sigma_1 \sigma_2 e^{K_0 L}}{\beta_1 e^{\alpha_2 Q} + \beta_2 e^{\alpha_1 Q}}$$

And

$$p(L) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/(2\sigma_1^2 \sigma_2^2)} \frac{e^{K_0 L}}{\beta_1 e^{\alpha_2 Q} + \beta_2 e^{\alpha_1 Q}}$$

5 Composite Distribution

Although the building type, traditional or thermally-efficient, may be inferred from information such as available databases or measurements, this may not always be possible. When the building type is unknown, a BEL PDF can be constructed from a universe of buildings that consists of traditional and thermally-efficient construction by creating a mixture distribution. If the fraction of traditional construction buildings is w_1 and the fraction of thermally-efficient buildings is w_2, then

$$p_0(L) = w_1 p_1(L) + w_2 p_2(L)$$

Where $p_0(L)$ is the composite PDF, $p_1(L)$ is the traditional construction PDF and $p_2(L)$ is the thermally-efficient PDF. Note since the weights w_1 and w_2 are fractions of the whole that:

$$0 \le w_1 \le 1 \qquad 0 \le w_2 \le 1 \qquad w_1 + w_2 = 1$$

As the CDF functions are merely the integrals of the PDF functions, the composite CDF function follows the same form.

$$P_0(L) = w_1 P_1(L) + w_2 P_2(L)$$

Where $P_0(L)$ is the composite CDF, $P_1(L)$ is the traditional construction CDF and $P_2(L)$ is the thermally-efficient CDF. When the building type is unknown, The FCC 20-51 Report and Order footnote 465 suggests the use of $w_1 = 0.7$ and $w_2 = 0.3$.

5.1 Mean and Variance

In general, the mean of a random variable is just its expected value and is given by:

$$\mu = \int_A^B x p(x)\, dx \quad \text{with} \quad \int_A^B p(x)\, dx = 1$$

Where $p(x)$ is the probability density function of x and is completely defined over the range A to B. The mean square of a random variable is the expected value of its square and is given by:

$$\sigma^2 + \mu^2 = \int_A^B x^2 p(x)\, dx$$

Where σ^2 is the variance (σ being the standard deviation) and μ^2 is the square of the mean. Since $p_0(L) = w_1 p_1(L) + w_2 p_2(L)$, the mean of the composite BEL is given by:

$$\mu_0 = \int_{-3}^{+\infty} L[w_1 p_1(L) + w_2 p_2(L)]\, dL = w_1 \int_{-3}^{+\infty} Lp_1(L)dL + w_2 \int_{-3}^{+\infty} Lp_2(L)dL = w_1\mu_1 + w_2\mu_2$$

Where μ_1 and μ_2 are the means of the traditional and thermally-efficient buildings, respectively. The composite mean square is given by:

$$\sigma_0^2 + \mu_0^2 = \int_{-3}^{+\infty} L^2 p_0(L)dL$$

$$= w_1 \int_{-3}^{+\infty} L^2 p_1(L)dL + w_2 \int_{-3}^{+\infty} L^2 p_2(L)dL = w_1(\sigma_1^2 + \mu_1^2) + w_2\,(\sigma_2^2 + \mu_2^2)$$

Where σ_1^2 and σ_2^2 are the variances of the traditional and thermally-efficient buildings, respectively. And consequently, the composite variance is:

$$\begin{aligned}\sigma_0^2 &= w_1(\sigma_1^2 + \mu_1^2) + w_2(\sigma_2^2 + \mu_2^2) - \mu_0^2 \\ &= w_1(\sigma_1^2 + \mu_1^2) + w_2(\sigma_2^2 + \mu_2^2) - (w_1\mu_1 + w_2\mu_2)^2 \\ &= w_1\sigma_1^2 + w_2\sigma_2^2 + w_1\mu_1^2(1 - w_1) + w_2\mu_2^2(1 - w_2) - 2w_1w_2\mu_1\mu_2 \\ &= w_1\sigma_1^2 + w_2\sigma_2^2 + w_1w_2\mu_1^2 + w_1w_2\mu_2^2 - 2w_1w_2\mu_1\mu_2 \\ &= w_1\sigma_1^2 + w_2\sigma_2^2 + w_1w_2(\mu_1 - \mu_2)^2\end{aligned}$$

6 Example Calculations

The following figures show the traditional, thermally-efficient and composite CDFs and PDFs at 6.5 GHz and an elevation angle of zero. The assumed weights for the composite functions are 70% traditional and 30% thermally-efficient construction.

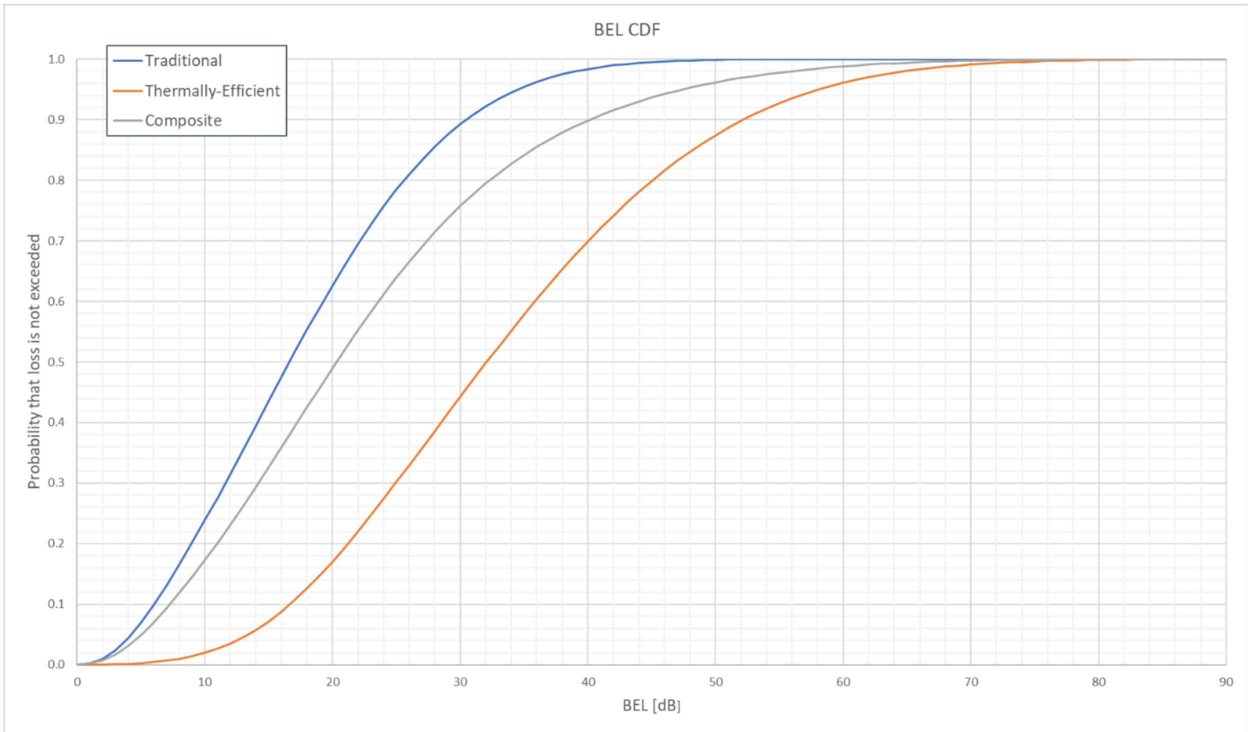

Figure 1 – CDF Frequency 6.5 GHz Elevation Angle 0 degrees 70/30 Composite

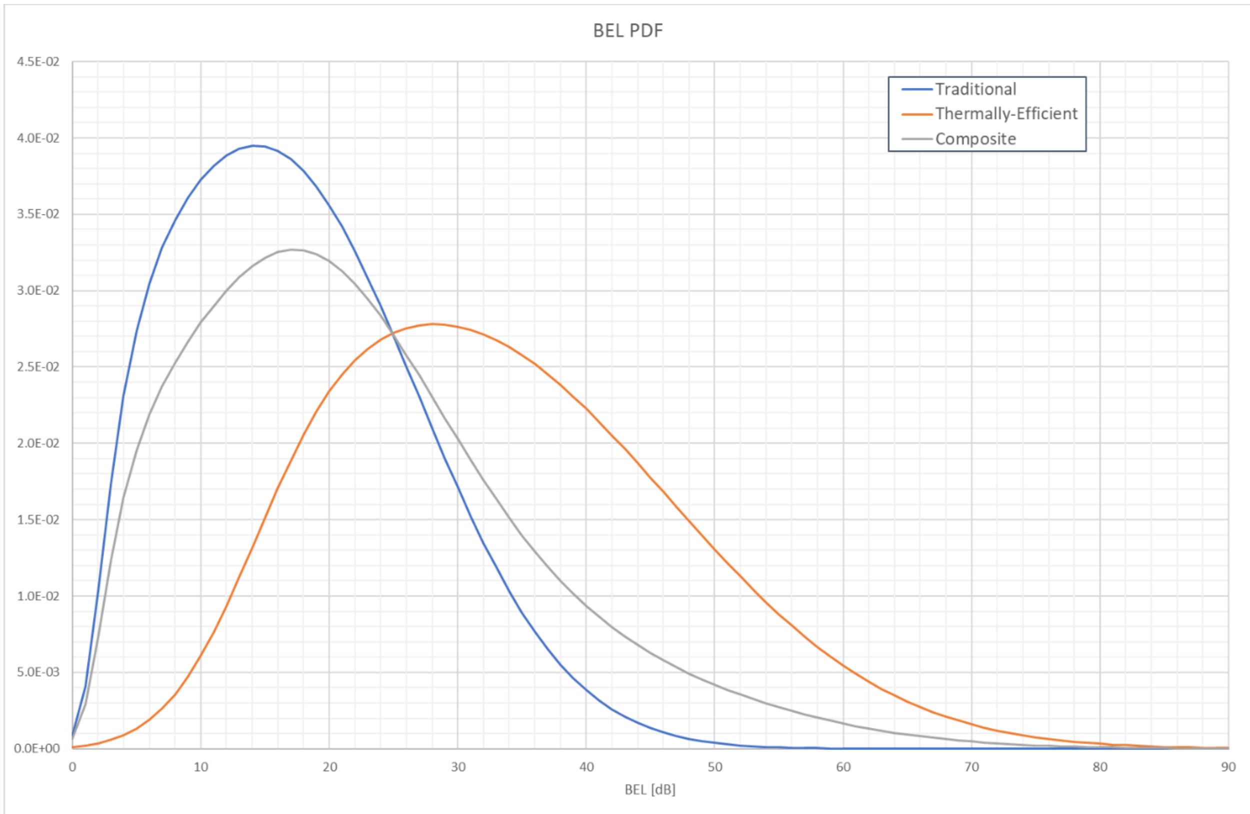

Figure 2 – PDF Frequency 6.5 GHz Elevation Angle 0 degrees 70/30 Composite

The following table list the means and standard deviations of the traditional, thermally-efficient and composite distributions.

Table 1 – BEL Statistics Frequency 6.5 GHz Elevation Angle 0 degrees

	Traditional	Thermally Efficient	Composite
weights	0.7	0.3	
mean [dB]	17.76	33.69	22.54
variance	89.33	193.97	174.01
standard deviation [dB]	9.45	13.93	13.19

7 Frequency Dependence

The P.2109 BEL probability distributions depend on the frequency of operation. As the UNII-5 through UNII-8 bands cover the range of frequencies from 5.925 GHz to 7.125 GHz, it is important to consider the differences in the distributions over these frequencies. In general, losses are smaller for the lower frequencies. Figure 3 and Figure 4 show the BEL CDF and PDF at 6.0 GHz, 6.5 GHz and 7.0 GHz for the traditional, thermally-efficient and composite distributions. The 6.5 GHz curves are shown by thicker lines which are flanked on either side by thinner lines for 6.0 GHz and 7.0 GHz. As can be seen in the figures, there is little change in the curves over the range of 6.0 GHz to 7.0 GHz. In fact, there is only a 0.32 dB difference in the median loss value of the composite distribution between 6.0 GHz and 7.0 GHz.

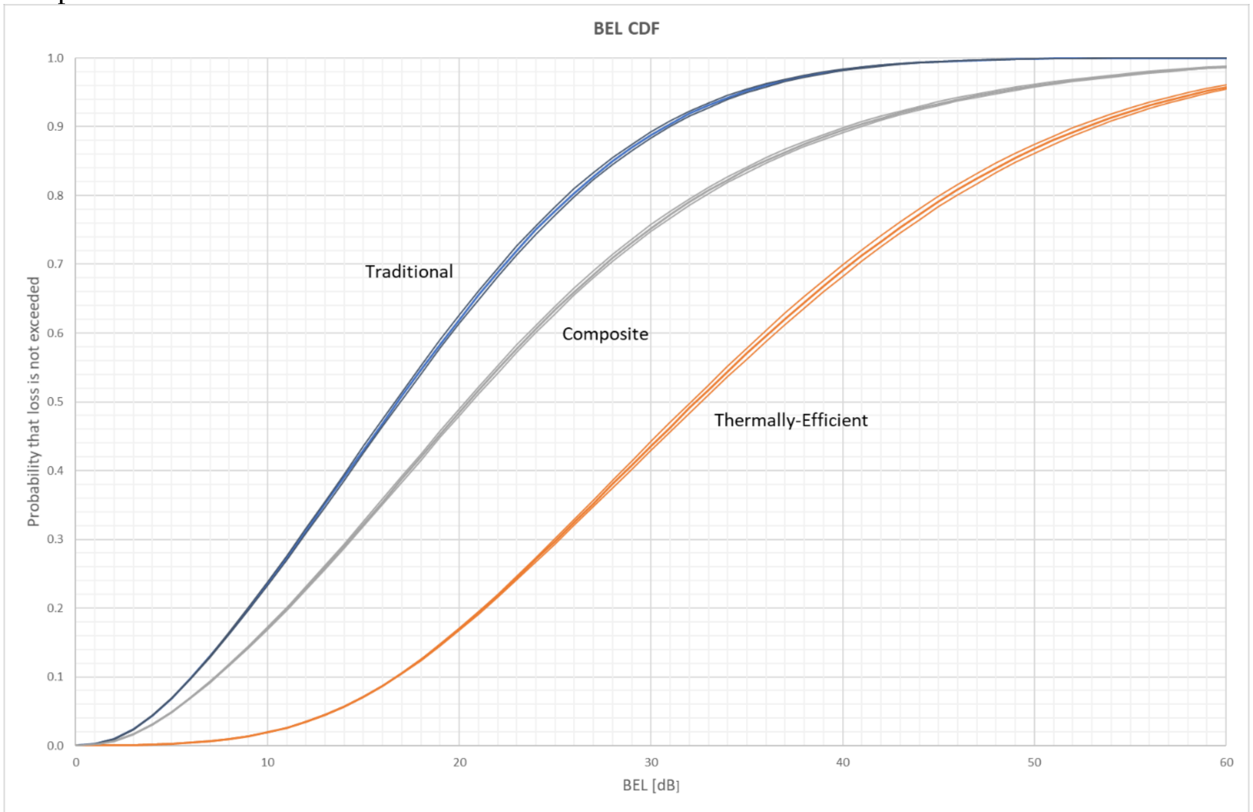

Figure 3 - BEL CDF at 6.0 GHz, 6.5 GHz and 7.0 GHz (0 degree elevation angle)

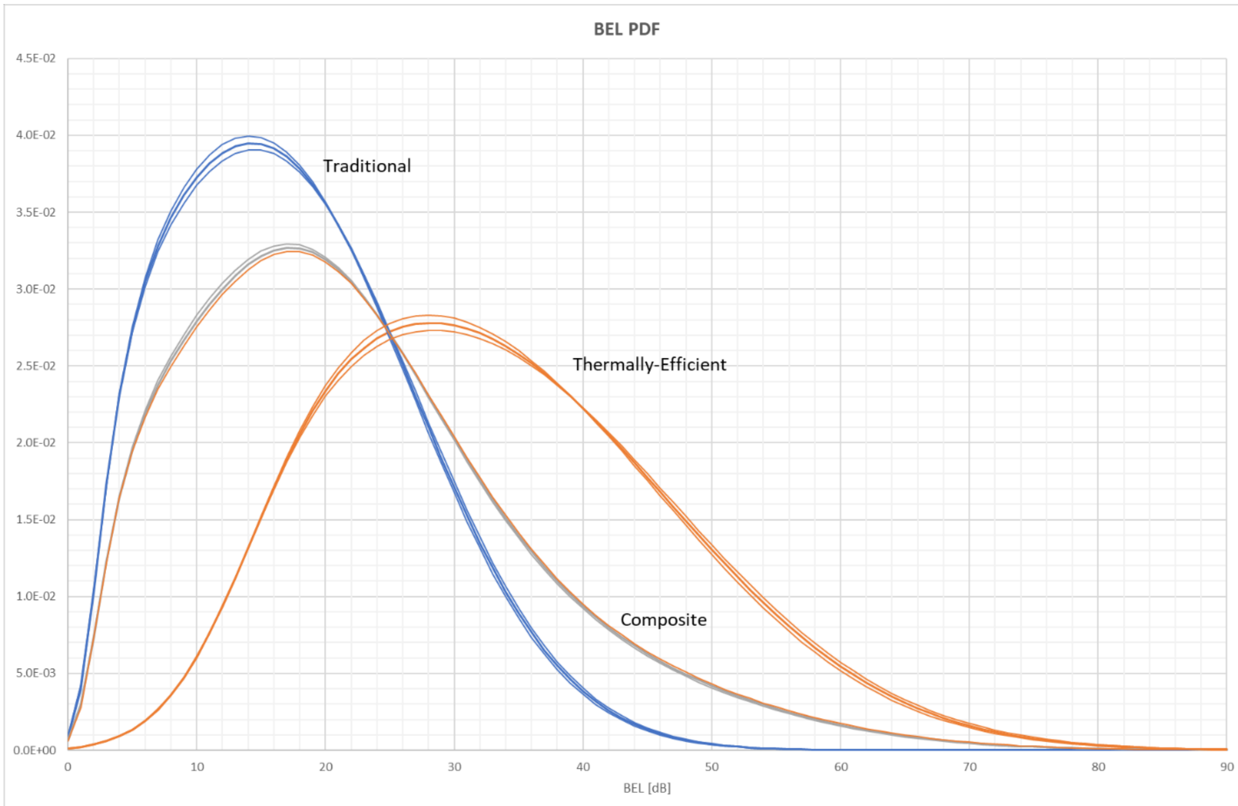

Figure 4 - BEL PDF at 6.0 GHz, 6.5 GHz and 7.0 GHz (0 degree elevation angle)

8 Elevation Angle Dependence

The elevation angle loss term from equation (10) in P.2109 results in predicting more loss for radiation entering or leaving a building at larger elevation angles. At 6.5 GHz, the increase in the median loss for traditional and thermally-efficient buildings is roughly a 1 dB increase for every 5 degrees of elevation angle. Figure 5 shows the 6.5 GHz BEL CDF curves for traditional and thermally-efficient buildings at elevation angles of 0, 5, 10, 15, 20, 30, 45 and 60 degrees. The traditional building curves are in blue, while the thermally-efficient building curves are in orange. Note that as the P.2109 recommendation is an empirical based model, some caution should be used with larger elevation angles where there are limited experimental results to base these predictions.

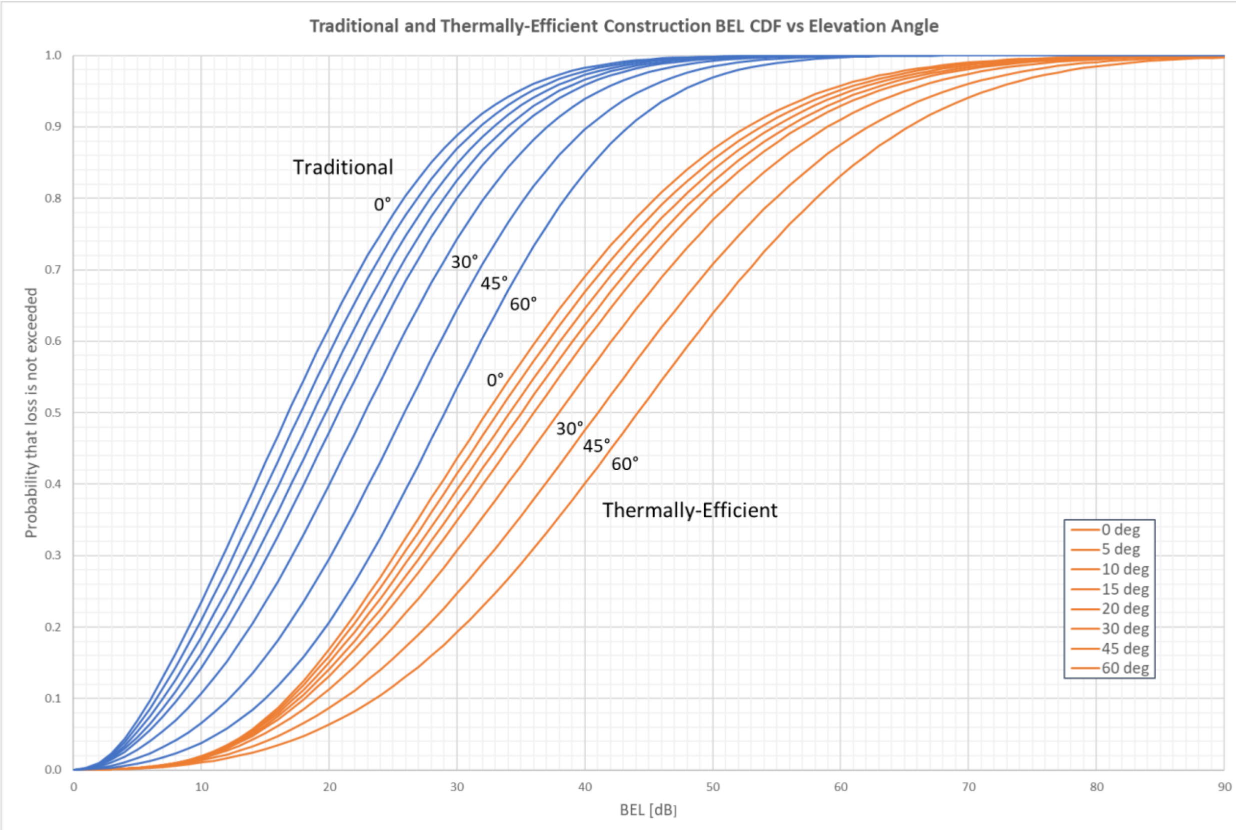

Figure 5 - BEL CDF at various elevation angles (6.5 GHz)

9 Random Number Generation

In running Monte-Carlo simulations the generation of random numbers that follow the traditional, thermally-efficient or composite BEL distributions can be useful. If a functional form of an inverse cumulative distribution is known, it can be used to produce random values that follow that distribution. This can be accomplished by starting with a uniform random number generator that produces values between zero and one. Applying the inverse cumulative distribution function to those values results in random numbers that following the desired distribution.

The inverse cumulative distribution for the BEL, $L(P)$, is precisely the function provided in equation (1) of P.2109. Consequently, random numbers that follow the traditional or thermally-efficient BEL distributions can be generated using equation (1) with the appropriate parameters and a uniform random number generator. For composite distributions, a collection of random numbers can be generated where some portion of them use the traditional construction parameters and the rest thermally-efficient construction parameters. For example, making a 70% traditional construction and 30% thermally-efficient construction composite, for every 10 random numbers generated 7 would use the traditional parameters and 3 the thermally-efficient parameters. The resulting set of random numbers would follow the 70/30 mixture distribution of the composite.